Tutorial Note IX

Exercise 0.1

Solve the following IBVP for the schrödinger equation:

$$\begin{cases} \mathrm{i}u_t = u_{xx} & (x,t) \in (0,1) \times \mathbb{R}^+; \\ u_x(0,t) = 0,\ u(1,t) = 0 & t \in \mathbb{R}^+; \\ u(x,0) = \varphi(x) & x \in [0,1]. \end{cases}$$

Proof. We first find the special solutions by separation of variables. Suppose that $u(x,t) = X(x)T(t)$. Then

$$\mathrm{i}X(x)T'(t) = X''(x)T(t),$$

and

$$\frac{X''}{X} = \frac{\mathrm{i}T'}{T} = -\lambda.$$

Moreover, X satisfies $X'(0) = 0$ and $X(1) = 0$. So $\lambda = [(n + 1/2)\pi]^2$ and $X_n(x) = \cos[(n+1/2)\pi x]$ where $n \geq 0$. Then $T_n(t) = \mathrm{e}^{i(n+1/2)^2 t}$. Decompose φ into a sum of X_n, that is,

$$\varphi(x) = \sum_{n=0}^{\infty} c_n \cos[(n + \tfrac{1}{2})\pi x],$$

where

$$c_n = 2\int_0^1 \varphi(x) \cos[(n + \tfrac{1}{2})\pi x] \,\mathrm{d}x.$$

Then

$$u(x,t) = \sum_{n=0}^{\infty} c_n \mathrm{e}^{i(n+1/2)^2 t} \cos[(n + 1/2)\pi x]$$

is a solution. □

Remark 0.1

The method of separation of variables is flexible. Ir could be applied to various equations and boundary conditions, and Cauchy problems. For example, the following two problems could be solved by the method of separation of variables:

$$\begin{cases} u_{tt} = u_{xx} - ru_t & (x,t) \in (0,1) \times \mathbb{R}^+; \\ u(0,t) = 0,\ u(1,t) = 0 & t \in \mathbb{R}^+; \\ u(x,0) = \varphi(x) & x \in [0,1]; \end{cases}$$

$$\begin{cases} u_{tt} = u_{xx} & (x,t) \in (0,1) \times \mathbb{R}^+; \\ u(0,t) = 0,\ (u_{tt} + u_x)(1,t) = 0 & t \in \mathbb{R}^+; \\ u(x,0) = \varphi(x) & x \in [0,1]. \end{cases}$$

Exercise 0.2

Show that the following IBVP has a non-zero solution:

$$\begin{cases} tu_t = u_{xx} + 2u & (x,t) \in (0,\pi) \times \mathbb{R}^+; \\ u(0,t) = 0,\ u(\pi,t) = 0 & t \in \mathbb{R}^+; \\ u(x,0) = 0 & x \in [0,\pi]. \end{cases}$$

Proof. We use the method of separation of variables to find a non-zero solution. Suppose that a solution has a form $X(x)T(t)$. Then

$$tX(x)T'(t) = X''(x)T(t) + 2X(x)T(t),$$

and

$$\frac{tT'}{T} = \frac{X'' + 2X}{X} = \lambda.$$

By solving

$$\begin{cases} X'' + (2-\lambda)X = 0; \\ X(0) = 0,\ X(\pi) = 0, \end{cases}$$

we have $\lambda = 2 - n^2$ and $X_n(x) = \sin nx$ for $n \geq 1$. We just consider X_1 here. Then $T(t)$ satisfies

$$tT'(t) = T(t),$$

and

$$T(t) = t$$

satisfies it and $T(0) = 0$. Now it is easy to check that $u(x,t) = t \sin x$ is a non-zero solution to the problem. □

Exercise 0.3

Solve the Klein-Gordon equation:

$$u_{tt} - u_{xx} + u = 0.$$

Proof. Here we use the idea of separation of variables to transform it into a 2D wave equation. By multiplying the two sides by $v(y)$, we have

$$[u(x,t)v(y)]_{tt} - [u(x,t(y)]_{xx} + u(x,t)v(y) = 0.$$

We want make

$$u(x,t)v(y) = -[u(x,t)v(y)]_{yy}.$$

So it suffices to let $v''(y) = -v(y)$. We could let $v(y) = \mathrm{e}^{\mathrm{i}y}$. Then $w(x,y,t) = \mathrm{e}^{\mathrm{i}y}u(x,t)$ solves the 2D wave equation:

$$w_{tt} - w_{xx} - w_{yy} = 0.$$ □